SPACE FIGURES

So far, we've been looking at two-dimensional geometry. If we got to three dimensions, we're looking at figures in **space.** Space is another undefined geometric term. In 2-D, the figures (lines, angles, polygons, etc.) all occur in a plane. In 3-D, there are an infinite number of planes. Each plane partitions space into three disjoint sets—the points on the plane and two half-spaces. Any two planes are either parallel (never cross, just like lines) or they intersect in a line. In figure 1, planes P and Q (blue) are parallel. Plane R (pink) intersects plane P at line l and plane Q at line m.

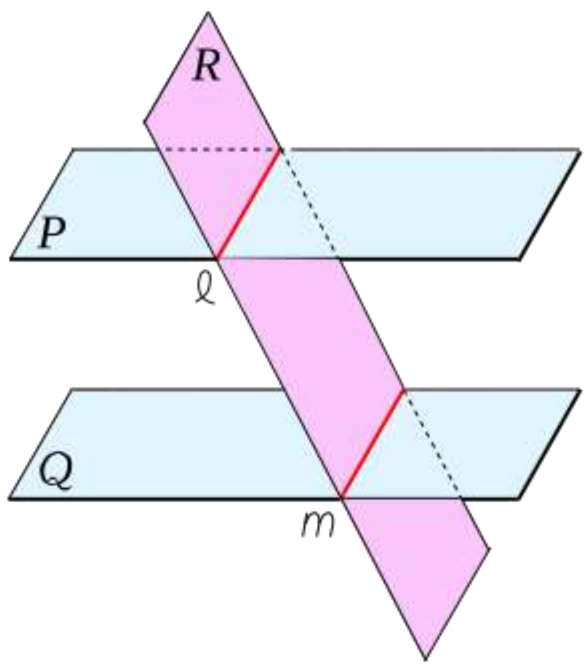

Figure 1: Parallel and Intersecting Planes
https://commons.wikimedia.org/wiki/File:Parallel_2_planes_and_Intersection.svg

When two planes intersect, we call the angle between the planes a **dihedral angle**. A dihedral angle is measured by measuring the angle whose sides lie in the planes and are perpendicular to the line of intersection of the two planes. Like any other angle, dihedral angles can be acute, right, or obtuse.

The surface of a space figure whose sides are polygonal regions is a **polyhedron** (plural is **polyhedra**). The polygonal regions are **faces**, and they intersect in the **edges** and **vertices** of the polyhedron. Remember that polygons can't include curved sides—so a space figure with curved sides, or a hole in a face, is NOT a polyhedron. Like polygons, polyhedra can be convex (no dent; line segment connecting two points stays inside the polyhedron) or concave (indentation; line segment between two points goes outside the polyhedron).

Regular Polyhedra

The best know polyhedra are the **regular polyhedra**, or **Platonic solids.** A regular polyhedral is a convex polyhedron who faces are congruent regular polygons, the same number of which meet at each vertex. The ancient Greeks proved that there were only five regular polyhedra. These shapes have all been found in nature: the tetrahedron, cube and octahedron occur as crystals; biologists have found fossils that are dodecahedrons and viruses that are icosahedrons. See the table below.

Polyhedra	Face Type	# of Faces	# of Edges	# of Vertices
Tetrahedron	Equilateral Triangle (3 per vertex)	4	6	4
Hexahedron (Cube)	Square (3 per vertex)	6	12	8
Octahedron	Equilateral Triangle (4 per vertex)	8	12	6
Dodecahedron	Regular pentagon (3 per vertex)	12	30	20
Icosahedron	Equilateral Triangle (5 per vertex)	20	30	12

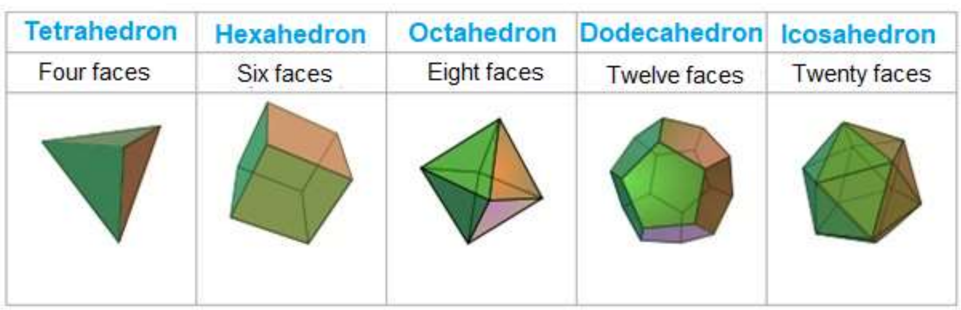

Figure 2: Platonic Solids
https://commons.wikimedia.org/wiki/File:Trupat_Platonik.jpg; labels changed to English
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Why are we so sure there's only five of these? Start with the smallest regular polygon: the equilateral triangle. There are 60° in each vertex. So we can fit 3 (180°), 4 (240°), or 5 (300°) in a vertex of a polyhedron, but 6 (360°) would lie flat. (360° or greater would no longer make a polyhedron.) The next regular polygon is a square, with 90° per vertex. We can fit 3 (270°) in the vertex of a polyhedron, but not 4 (360°). A regular pentagon has 108° per vertex, so again we can fit 3 (324°) but not 4 (432°) in the vertex of a polyhedron. Next up is a regular hexagon, which has 120° per vertex, but 3 (360°) won't fit in the vertex of a polyhedron. Any larger regular polygon would have the same problem. So these are the only 5 regular polyhedra.

Nets

A two-dimensional shape that can be folded into a three-dimensional shape is called a **net**. Being able to visualize the net for a 3-D shape is important when trying to calculate volume and surface area. Can you visualize the nets for a cube? There are actually **11** distinct nets that fold into a cube. Visit Cube Nets (Illuminations) to practice identifying the nets for a cube.

Activity: Nets for Three-Dimensional Figures

Materials: 2-centimeter grid paper (or other large-scale grid) and scissors. Tape optional.

1. A cube can be formed by creasing and folding along the lines of the pattern shown here. Use your grid paper to form and cut out several different patterns that will fold into a cube *with no overlaps*. These patterns are called **nets**; there are 11 distinct patterns that will form a cube. Record sketches of your patterns.

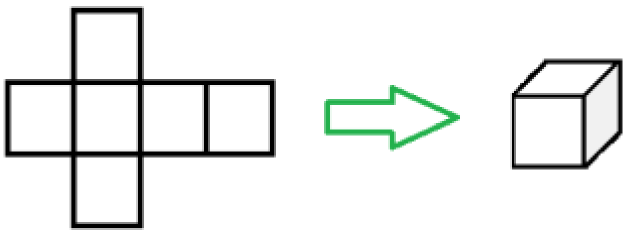


Figure 3: Cube Net

2. Form and cut out a net that will fold into the two-cube stack shown here. Note that there is no overlap, and no squares hidden inside the cube (the inside is hollow). Sketch your net.

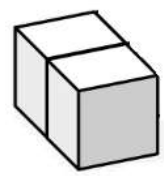


Figure 4: Two Cubes

3. Visualize a stack of *n* cubes, and describe a set of nets that will fold and cover this stack. Write an algebraic expression for the number of squares in this net.

4. Select two of the following figures and form their nets on grid paper. Show sketches of your nets. The number of cubes in the figure is the **volume** of the figure, and the number of squares in the net is the **surface area** of the figure. Determine the volume and surface area of each figure you select.

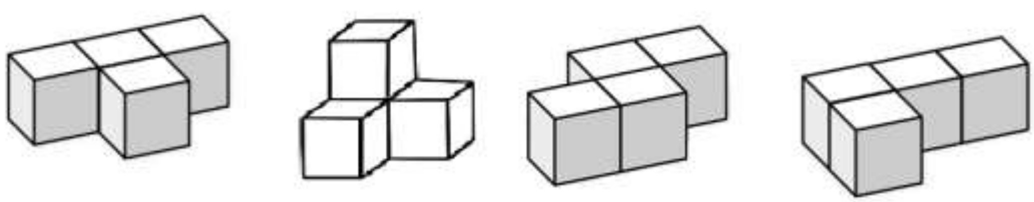


Figure 5: Cubic Boxes
https://www.rawpixel.com/image/8703960/image-pattern-illustrations-public-domain Combined into one image

(Activity expanded from Mathematics for Elementary Teachers: A Conceptual Approach, McGraw Hill Higher Education, 2016.)

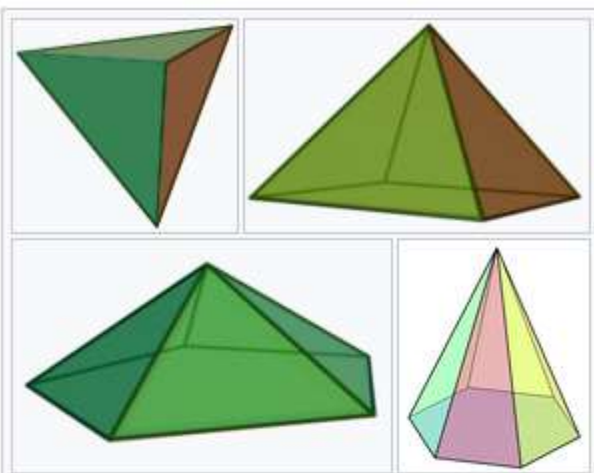
Classifying Space Figures

We're going to look at the following space figures: pyramids, prisms, cones, cylinders, spheres. Note that these aren't all polyhedra, because they have faces that aren't polygons (like circles). Before you go on, can you define these space figures?

Pyramids

You probably think of the pyramids in Egypt when you hear this word. The Egyptian pyramids have a square base, and triangular sides rising up to the vertex; this is a **square pyramid.** In general, the base of a pyramid can be any polygon (they're named according to the polygon in their base), but the sides are always triangular.

The vertex not contained in the pyramid's base is called its **apex.** Pyramids whose sides are isosceles triangles are **right pyramids**; the apex is centered over the base (all the pyramids in figure 6 are right pyramids). Otherwise, the pyramid is an **oblique pyramid**; the triangular sides are not isosceles.

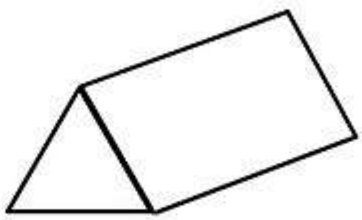
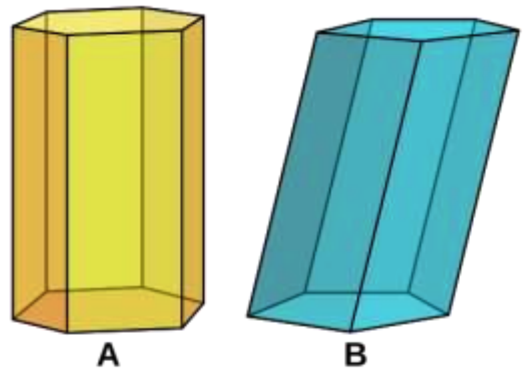


The family of a regular polygonal base pyramid: tetrahedron, square pyramid, pentagonal pyramid, and hexagonal pyramid.

Figure 6: Pyramids
https://en.wikipedia.org/wiki/Pyramid_(geometry)

Prisms

You might remember using a prism in science class to split light into a spectrum of colors, using a triangular prism. A **prism** has two parallel bases, upper and lower, which are congruent polygons; the sides are parallelograms. Like pyramids, prisms are named by the shape of their bases. The most common prism is a rectangular prism—most boxes are rectangular prisms. If the sides of the prism are perpendicular to the bases, they're rectangles; this makes a **right prism.** If some of the faces of the prism are not rectangles, it's an **oblique prism.**

 *Figure 7: Triangular Prism* *https://commons.wikimedia.org/wiki/File:Triangular_Prism_Drawing.jpg* *CC BY-SA 4.0*	 *Figure 8: Right and Oblique Prisms* *https://commons.wikimedia.org/wiki/File:Right_and_not-right_prism.svg* *CC BY-SA 3.0*

Cones and Cylinders

Cones and cylinders are the circular counterparts of pyramids and prisms. Note, however, that they are NOT pyramids or prisms, because their sides aren't triangles or parallelograms.

A **cone** has a circular region (disk) for a base and a lateral surface that slopes to the vertex/apex. Ice cream cones, paper cups, and party hats are common examples of cones. If the vertex is directly over the center of the base, the cone is a **right cone**; otherwise it's an **oblique cone**. Technically, the base can be any simple closed curve, but circles are the most common.

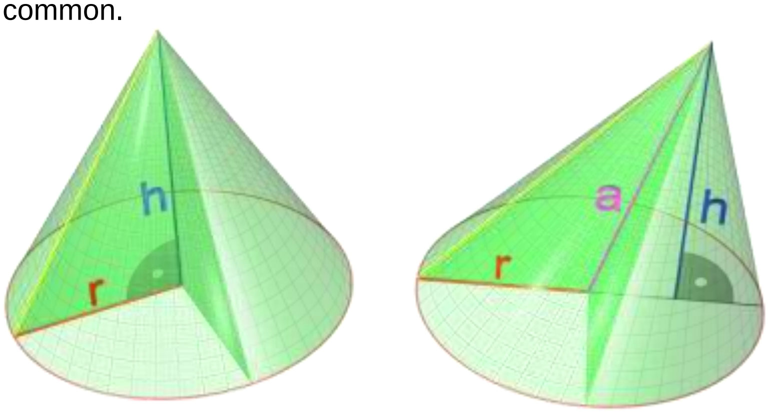


Figure 9: Right cone and oblique cone
https://en.wikipedia.org/wiki/Cone#/media/File:Cone_3d.png
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Ordinary cans are models of cylinders. A **cylinder** has two parallel circular bases (disks) of the same size, and a surface a lateral surface that rises from one base to the other. Technically, the base can be any simple closed curve, but circles are the most common. If the center of the upper base and lower base line on a line perpendicular to each base, it's a **right cylinder**; otherwise, it's an **oblique cylinder.** Oblique cones and cylinders are not very common.

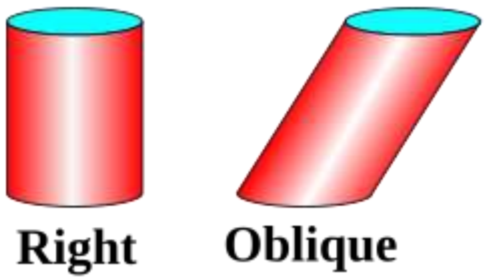

Figure 10: Right and oblique cylinders
https://commons.wikimedia.org/wiki/File:Cylinders.svg

Spheres

A **sphere** is the set of points in space that are the same distance from a fixed point, called the **center.** Globes and round balls are common examples of spheres. A line segment joining the center of the sphere to a point on the sphere is a **radius;** the length of this segment is also called the radius of the sphere. A line segment connecting points on the sphere and passing through the center is a **diameter**; the length of this line is also called the diameter of the sphere.

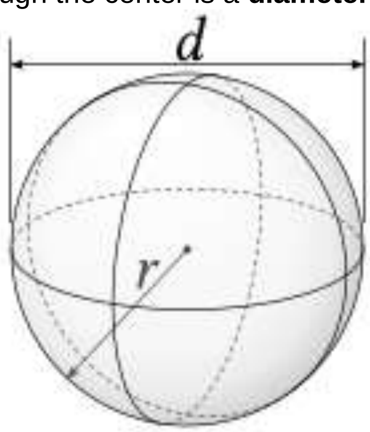

Figure 11: Sphere
https://commons.wikimedia.org/wiki/File:Sphere_(parameters_r,_d).svg