# IRRATIONAL NUMBERS

Scientists like to classify things, and mathematicians are no exceptions. We've already talked about several different kinds of numbers.

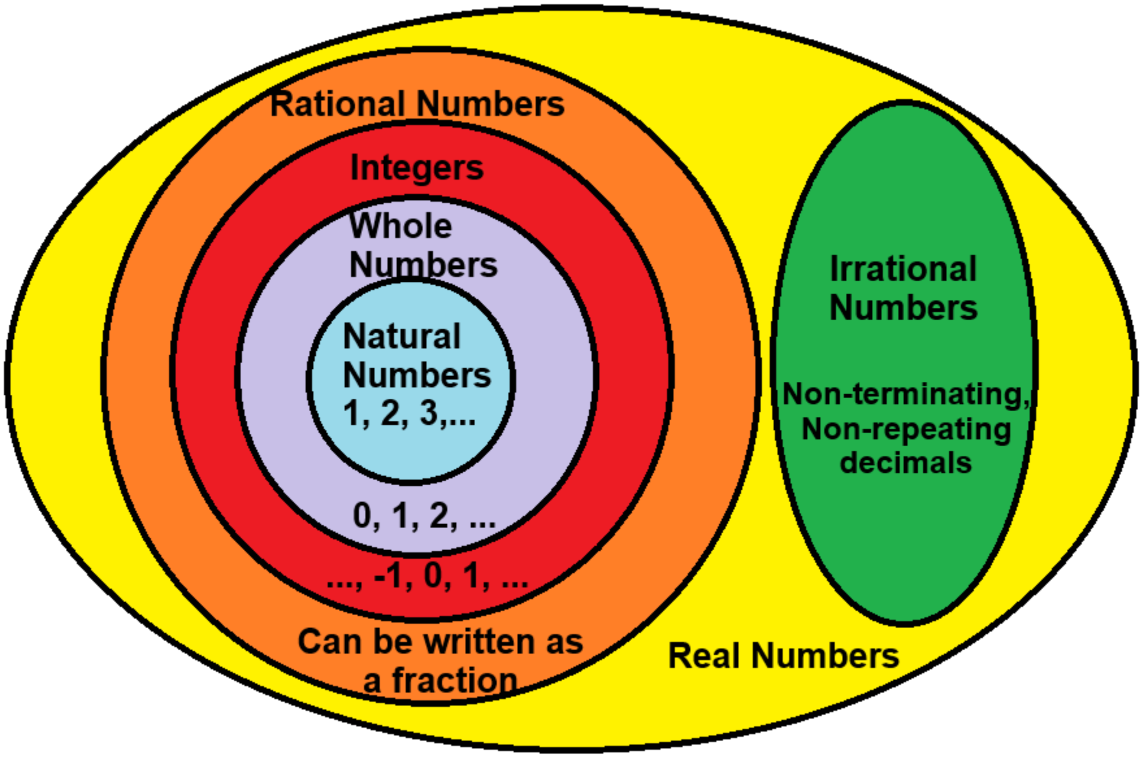


*Figure 1: Number Classifications*

The counting numbers are called **natural numbers**. When we add zero, we have **whole numbers**. When we add negatives, we get **integers**. Anything that can be written as a fraction is a **rational number**. This includes terminating decimals (ones that end) and repeating decimals. (See the conversion section for more on repeating decimals.) Note that these types of numbers are nested; stuff in an inner circle can also be classified by a larger circle it's contained in. 3 is a natural number, a whole number, an integer, a rational number, and a real number. -7 is an integer, a rational number, and a real number. It doesn't necessarily go the other direction, however; 1/2 is a rational number but not an integer.

An **irrational number** is a non-terminating, non-repeating decimal. Ancient people used to believe only rational numbers existing, but it's easy to come up with a real-life example of a rational number; that's the point of the activity below. Some irrational numbers you may have encountered include $\pi$, $e$, $\sqrt{2}$, etc. Any square root that doesn't come out to a whole number is an irrational number.

If a number is either rational or irrational, it's classified as a **real number**. There are non-real numbers, but most students don't encounter those until high school. For this class, we will always be working in the real number system.

# Irrational Numbers

The discover of irrational numbers is typically attributed to the Pythagoreans in the fifth century BCE. If you have a right triangle with sides length 1, the hypotenuse has length $\sqrt{2}$.

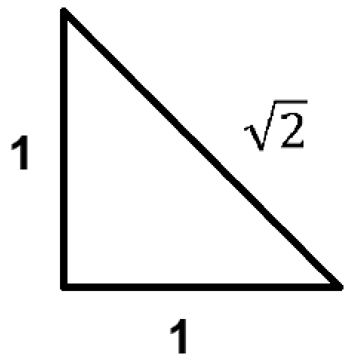


*Figure 2: Triangle with hypotenuse sqrt(2)*

## Activity: Irrational Numbers on the Geoboard

The point of this activity is that you can have real-life, tangible lengths that are irrational numbers.

A geoboard allows students to create and explore shapes using a series of pegs and rubber bands (see image below). Virtual versions exist, too. Virtual geoboard images from this section come from https://mathsbot.com/manipulatives/geoboard .

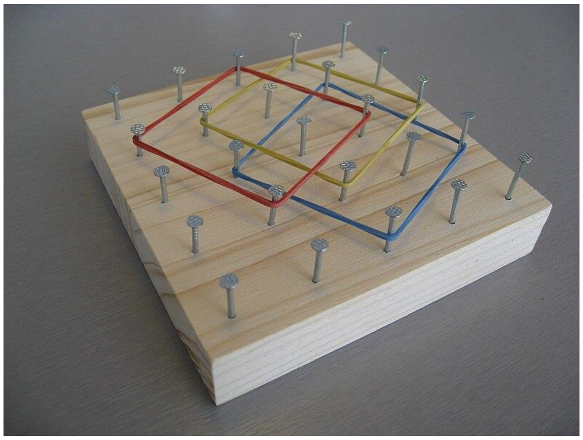

*Figure 3: Geoboard*
*https://commons.wikimedia.org/wiki/File:Quadrate_auf_dem_Geobrett_Schr%C3%A4gsicht.jpg*

Area on a geoboard is determined by counting the number of squares inside the outline. In the image below, the green square has an area of 9 square units. The blue square has an area of 9 square units, too. The red square has an area of 5 square units. You can see this by noting that each of the triangles defined by 2 blue lines and one red one has an area of 1 square unit $\left(\frac{1}{2}bh = \frac{1}{2}(1)(2) = 1\right)$, and 9-4=5.

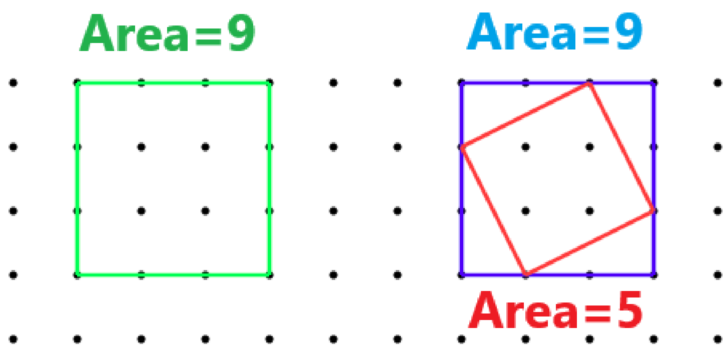


*Figure 4: 3 geoboard squares*

Squares can be formed on the geoboard having areas of 1, 2, 4, 5, 8, 9, 10, and 16 square units. Use geoboard paper or a virtual manipulative to see if you can find each of these squares.

***So what?*** If your square has area of 5 square units, that means the length of each side is $\sqrt{5}$ units. A real-life example of a distance that is an irrational number.

## Pythagorean Theorem

One of the first places students encounter irrational numbers is in the context of using the Pythagorean Theorem to solve for sides of triangles. A lot of people remember the formula

$$a^2 + b^2 = c^2$$

but not what it means. *a* and *b* are the legs of a right triangle (a triangle with a 90° angle), and *c* is the hypotenuse. It's important to note that the hypotenuse is the longest side, which is also the side opposite the right angle.

*The Pythagorean Proposition* (E. S. Loomis, 1968) is a book that describes 370 proofs of the Pythagorean Theorem. There are many visual and cut-and-paste proofs available online. It's a good idea to look at several to see how they work.

Note that it's possible to solve for *any* side of a right triangle using the Pythagorean Theorem. You're not always solving for c. See two examples below. Also note that if your squared term isn't a perfect square, you're going to get an irrational answer. We will discuss simplifying square roots after the examples.

Pyth 1:

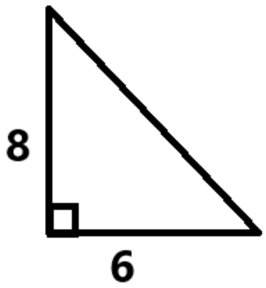


*Figure 5: Right triangle with legs 6 and 8*

$$6^2 + 8^2 = c^2$$
$$36 + 64 = c^2$$
$$100 = c^2$$
$$10 = c$$

Pyth 2:

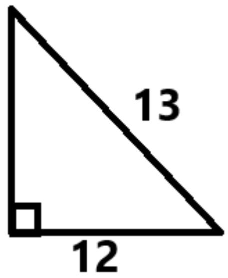


*Figure 6: Right triangle with leg 12 and hypotenuse 13*

$$a^2 + 12^2 = 13^2$$
$$a^2 + 144 = 169$$
$$a^2 = 25$$
$$a = 5$$

## Simplifying Square Roots

You should always simplify square roots as much as possible. This means removing any perfect squares from the radical. (Yes, you can approximate it with a calculator—but these are irrational numbers, so the only way to be exact is to keep the radical in your answer.)

For example:

$$\sqrt{300} = \sqrt{100 \times 3} = 10\sqrt{3}$$

If you don't all the squares out the first time, you can do it in several steps:

$$\sqrt{300} = \sqrt{25 \times 12} = 5\sqrt{12} = 5\sqrt{4 \times 3} = 10\sqrt{3}$$

Note that you end up in the same place.